MATH PRACTICE STANDARDS

Make sense of problems and persevere in solving them.

Mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. Mathematically proficient students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. Younger students might rely on using concrete objects or pictures to help conceptualize and solve a problem. Mathematically proficient students check their answers to problems using a different method, and they continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

Reason abstractly and quantitatively

Mathematically proficient students make sense of quantities and their relationships in problem situations. They bring two complementary abilities to bear on problems involving quantitative relationships: the ability to *decontextualize*—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to *contextualize*, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

Construct viable arguments and critique the reasoning of others.

Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively about data, making plausible arguments that take into account the context from which the data arose. Mathematically proficient students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. Elementary students can construct arguments using concrete referents such as objects, drawings, diagrams, and actions. Such arguments can make sense and be correct, even though they are not generalized or made formal until later grades. Later, students learn to determine domains to which an argument applies. Students at all grades can listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Use appropriate tools strategically.

Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. Proficient students are sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, mathematically proficient high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. Mathematically proficient students at various grade levels are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

Attend to precision.

Mathematically proficient students try to communicate precisely to others. They try to use clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, including using the equal sign consistently and appropriately. They are careful about specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. In the elementary grades, students give carefully formulated explanations to each other. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

Look for and make use of structure.

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see 7×8 equals the well remembered $7 \times 5 + 7 \times 3$, in preparation for learning about the distributive property. In the expression $x2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y.

Look for and express regularity in reasoning.

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through (1, 2) with slope 3, middle school students might abstract the equation $(y - 2)/(x - 1) = 3$. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x2 + x + 1)$, and $(x - 1)(x3 + x2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

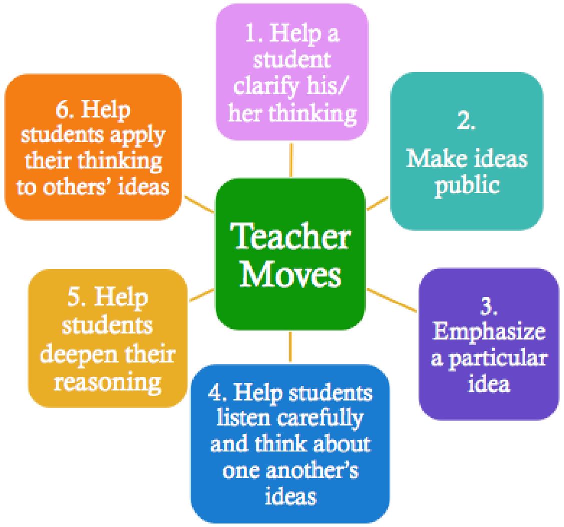
Teaching Practice	Teacher Actions	Student Actions
Establish math goals to focus learning	Teachers help students understand how the current work contributes to their learning by discussing and referring to the purpose and goal of the lesson.	Students are able to discuss what they're working to learn and why they're learning it.
Implement tasks that promote reasoning and problem-solving	Teachers develop high cognitive demand tasks that provide multiple entry points.	Students accept responsibility for making sense of tasks, and persevere in exploring and reasoning through tasks.
Use and connect mathematical representations	Teachers ask students to draw something that represents their thinking, introduce other forms of representations, and help students discover the connections among various types of representations.	Students sketch diagrams to help them make sense of problems, and describe and justify their reasoning with drawings, diagrams, and other representations.
Facilitate meaningful math discourse	Teachers position students as authors of ideas, who explain and defend their approaches.	Students present and explain ideas to one another, and ask one another clarifying questions, try out one another's strategies, and describe the approaches used by others.
Pose purposeful questions	Teachers advance student thinking by asking questions that carefully elicit and build on (but do not take over or 'funnel') student thinking.	Students expect to be asked, and so are prepared to, explain and elaborate on their reasoning.
Build procedural fluency from conceptual understanding	Teachers ask students to explain why the procedures they've chosen work to solve particular problems.	Students are able to use multiple strategies to solve problems, and can explain the mathematical basis for the strategies they're using.
Support productive struggle	Teachers praise students for their efforts ad perseverance, and support them by asking questions that scaffold their thinking without stepping in to provide answers.	Students acknowledge their confusion without giving up on solving problems, and ask questions to clarify their thinking and approaches.
Elicit and use evidence of student thinking	Teachers elicit and gather evidence of student thinking at strategic points during instruction.	Students ask questions, respond to and give suggestions to one another to support the collaborative development of understanding.

BELIEFS ABOUT TEACHING AND LEARNING MATHEMATICS

Math learning should focus on practicing procedures and memorizing basic number combinations.	Math learning should focus on developing understanding of concepts and procedures through problem-solving, reasoning, and discourse.
Students need only to learn and use the same standard computational algorithms and the same prescribed methods to solve algebraic problems.	All students need to have a range of strategies and approaches from which to choose in solving problems, including, but not limited to, general methods, standard algorithms, and procedures.
Students can learn to apply math only after they have learned the basic skills.	Students can learn mathematics through exploring and solving contextual and mathematical problems.
The role of the teachers is to tell students exactly what definitions, formulas, and rules they should know and demonstrate how to use this information to solve problems.	The role of the teacher is to engage students in tasks that promote reasoning and problem-solving and facilitate discourse that moves students toward shared understanding of mathematics.
The role of the student is to memorize information that is presented and then use it to solve routine problems on homework, quizzes, and tests.	The role of the student is to be actively involved in making sense of math tasks by using varied strategies and representations, justifying solutions, making connections to prior knowledge or familiar contexts and experiences, and considering the reasoning of others.
An effective teacher makes the math easy for students by guiding them step-by-step through problem-solving to ensure that they are not frustrated or confused.	An effective teacher provides students with appropriate challenges, encourages perseverance in solving problems, and supports productive struggle in learning mathematics.

Level	Teacher Role	Questioning	Explaining Mathematical Thinking	Building Student Responsibility within the Community
0	Teacher is at the front of the room and dominates the conversation.	Teacher is the only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher questions focus on correctness. Students provide short, answer-focused responses. Teacher may provide answers.	Classroom culture supports students keeping their ideas to themselves or responding only when asked.
1	Teacher encourages the sharing of math ideas and directs speaker to talk to the class, not to the teacher only.	Teacher questions begin to focus on student thinking and less on answers. Only the teacher asks questions.	Teacher probes student thinking somewhat. Students provide brief descriptions of their thinking in response to probing. One or two strategies may be elicited. Teacher may fill in a sparse explanation.	Students believe that their ideas are accepted by the classroom community. They begin to listen to one another supportively and to restate in t heir own words what a classmate has said.
2	Teacher facilitates conversation between students, and encourages students to ask questions of one another.	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with prompting from the teacher.	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to probing and volunteer their ideas. Students begin to defend their answers.	Students believe that they are math learners and that their ideas and others' ideas are important. They listen actively so they can contribute.
3	Students carry the conversation themselves. Teacher guides from the periphery. Teacher waits for students to clarify the thinking of others.	Student-to-student talk is initiated by students. Students ask questions and listen to responses. Many questions ask 'why' and call for justification. Teacher questions may still guide the discourse.	Teacher follows students' explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from teacher.	Students believe that they are math leaders and can shape the thinking of others. They help shape the class thinking in supportive, collegial ways and accept the same support from others.

Teacher Discourse Moves



1. Help a student clarify his/her thinking

Wait time: 20-30 seconds after questions and after responses.

"Can you say some more about that?"

"Can you show us what you mean?"

"Can you draw that?"

2. Make ideas and thinking public and available for discussion

"Tell us more about what you're thinking."

Clarify/repair how idea is expressed, without overriding student's ownership.
"Did I say your idea correctly?"

Re-voice to connect everyday expression to more precise academic language. *"So, you're saying…"*

3. Mark/emphasize a particular idea.

"Rebroadcast" an idea by revoicing, or ask a student to re-voice or paraphrase to give an idea more exposure so everyone can hear it and think about it again.

"That's interesting. Can you say that again for us?"

"Will someone re-tell that idea for us?"

4. Help students listen carefully to and think about others' ideas

"Who can rephrase or repeat that idea for us?"

"How is that idea different from what we had said earlier?"

"Who wants to explain the evidence that Group A used?"

"Do you agree or disagree with that?"

"Whose idea/thinking is most different from your own?"

5. Help students deepen their reasoning

"Will you tell us more about your thinking on that? Why do you think that works?"

"Would that always be true?" "Is there a condition that would make that false?"

"How could you show that that is true?"

"How could we revise our model to account for this?"

"What new questions do you have now? What do we need to know more about now?"

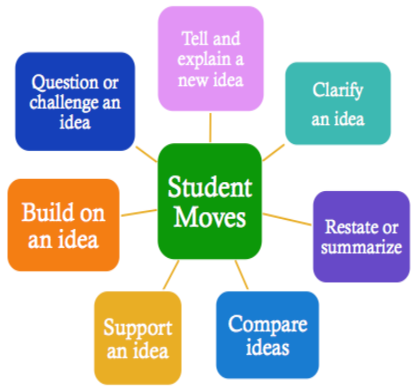
6. Help students apply their thinking to others' ideas; prompt peer-to-peer talk

"Who will re-tell that idea for us? Please check back with X to see if you told it correctly."

"Who is ready to tell us the connection between those two ideas?"

"You look uncertain. What can you ask X to find out more?"

"How does that idea build on the last one? What's the connection?"

Student Idea Moves	1. Tell or explain a new idea	2. Clarify someone's idea	3. Restate or summarize an idea.
	"I think…" "I know it will work because…" "The best strategy would be…."	"Say again, please." "What did you mean when you said…?" "Are you saying that…?"	"He said…" "In other words, …" "The suggestion was made that…"
4. Compare ideas	**5. Support an idea**	**6. Build on an idea**	**7. Question or challenge an idea**
"The same." "Ours is better because…" "The new strategy is more efficient because…"	"Good idea, because…" "Remember, it said in our book that…" "The advantage of that method is…"	"Let's try that." "We should change our model to show that." "That idea would help us figure out why…"	"Why?" "But that doesn't explain what we saw when…" "Is there a more efficient way to …?"