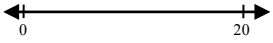
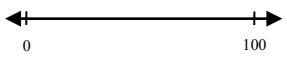
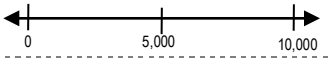
Developmental Guidelines

Developmental Guidelines

Number Development – Whole Numbers					
	Level 0	**Level 1**	**Level 2**	**Level 3**	**Level 4**
Common Core	Pre-K	Pre-K/K	Pre-K/K	K/1	1/2
Forward Number Word Sequence	May be able to count to 10 or beyond but does not have 1:1 correspondence consistently	Can count 1-20 Can name the number directly after for numbers 1-20 but may have to count from 1 to do so.	Can count by 1s from any number 1-30; Can name number directly after for numbers 1-30	Can count by 1s from any number 1-100; Can name the number directly after for numbers 1-100	Can count by 1s from any number 1-200; Can name the number directly after for numbers 1-200
Backward Number Word Sequence		Can count back from 10; Can name the number directly before for numbers 1-5 but may have to count from 1	Can consistently count back within 10-0 (starting from any number); Can inconsistently count back from 30; Can inconsistently name the number directly before for numbers from 1-30 but may have to count from smaller number.	Can count back within 30-0 (starting from any number); Can name the number directly before for numbers 1-30	Can count back within 100-0 (starting from any number); Can name the number directly before for numbers 1-100
Number Word Sequences: Skip Counting				Can count forward and backward by 10s	Can count by 2s, 5s, and 10s starting from the first multiple
Number ID		Can identify numerals 1-10 consistently, 11-20 inconsistently	Can identify numerals 1-30 consistently Can write numerals 1-20	Can identify and write numerals 1-100	Can identify numerals 1-1000 Can write numerals 1-200
Sequencing/ Ordering		Can sequence numbers in the range 1-20	Can sequence numbers in the range 1-30	Can sequence and order numbers in the range 1-100	Can sequence and order numbers in the range 1-200
Number Line/ Magnitude					Can accurately place numbers 1-20 on an empty number line (Example: Where does 15 go?) 
Subitizing		Can subitize regular dot patterns to 6 Can subitize finger patterns to 5	Can subitize quantities in pair-wise and five-wise ten frames Can subitize finger patterns to 10		
Object Counting		Can count or make sets of up to 10 accurately.	Can count or make sets of up to 30 accurately.		Can count or make sets of up to 100 using base-ten materials (10s and 1s)

Developmental Guidelines

Number Development – Whole Numbers

	Level 5	Level 6	Levels 7-9	Level 10+
Common Core	1/2	3/4	4	4+
Forward Number Word Sequence	Can count by 1s from any number 1-1000; Can name the number directly after for numbers 1-1000	Can count by 1s from any number 1-10,000 Can name the number directly after for numbers 1-10,000	Can count forward and backward by 1s, 10s, 100s, 1,000s and 10,000s from any number up to 100,000	Can count forward and backward by 1s, 10s, 100s, and other powers of ten from any whole number
Backward Number Word Sequence	Can count back within 1000-0 (starting from any number); Can name the number directly before for numbers 1-1,000	Can count back within 10,000-0 (starting from any number); Can name the number directly before for numbers 1-10,000		
Number Word Sequences: Skip Counting	Can count by 2s, 3s, 4s, and 5s starting from any multiple Can count forward and backward by 10s from any 2-digit number	Can count forward and backward by 10s and 100s from any 2 or 3 digit number Can count forward and backward by 25s starting from any multiple		
Number ID	Can identify and write numerals 1-1000	Can identify and write numerals 1-10,000	Can identify and write numerals 1-100,000	Can identify and write numerals beyond 100,000
Sequencing/ Ordering	Can sequence and order numbers in the range 1-1000	Can sequence and order numbers in the range 1-10,000	Can sequence and order numbers in the range 1-100,000	Can sequence and order numbers beyond 100,000
Number Line/ Magnitude	Can accurately place numbers 1-100 on an empty or nearly empty number line (Example: Where does 65 go?) 	Can accurately place numbers 1-1000 on an empty or nearly empty number line (Example: Where does 215 go?) 	Can accurately place numbers 1-10,000 on an empty or nearly empty number line (Example: Where does 2,150 go?) 	Can accurately place whole numbers on empty or nearly empty number lines.
Subitizing				
Object Counting	Can count or make sets of up to 1,000 using base-ten materials (100s, 10s, and 1s)			

Developmental Guidelines

Operation Development – Addition and Subtraction with whole numbers

	Level 0	Level 1	Level 2	Level 3	Level 4
Common Core	Pre-K	K	K/1	K/1/2	1/2
Problem Types	Solves problems embedded in daily routines **using direct modeling** including Add to-Result Unknown and Take from-Result Unknown contexts with totals to 5 (See CCSSM, pg. 88 for descriptions of problem contexts)	Uses strategies described below to solve problems embedded in daily routines **using direct modeling** including Add to-Result Unknown and Take from-Result Unknown contexts with totals to 10	Uses strategies described below to solve Add to-Result Unknown and Take from-Result Unknown story problems with totals to 20 **using direct modeling**	Uses strategies described below to solve story problems with totals to at least 30, including: Add to: Result unknown and change unknown Take from: Result unknown and change unknown Put together/take apart: total unknown	Uses strategies described below to solve story problems with totals to 100, including: Add to: Result unknown and change unknown Take from: Result unknown and change unknown Put together/take apart: total unknown Compare: difference unknown
Strategies Supported by Materials		Can model problems using concrete objects, and count the objects to find the answer. Involving the child in acting out the problem may support the child's understanding of the problem.	Can solve a covered task when one or both sets are covered, the first number is within 1-20, the second number is within 1-5, and the total is within 1-20. Counts from 1 to do so.	Can count on to solve additive (e.g. 15 + 2 = ☐) and missing addend (e.g. 15 + ☐ = 17) covered tasks Can count back to solve removed item covered tasks (e.g. 17 – 2 = ☐) May be able to count up or back to solve missing subtrahend covered tasks (e.g. 17 – ☐ = 15)	Can count up or back to solve missing subtrahend covered tasks (e.g. 17 – ☐ = 15) Can increment by 10s and 1s when supported by base-10 models such as base-10 blocks, ten-frames, etc.
Counting Strategies			When solving joining problems, counts from 1, but may continue the count when counting the second set. (i.e, may count the first set from 1, then the second set from 1, then the joined set from 1; or may count the first set from 1, then continue by counting the second set to reach the total). When solving separating problems, counts forward three times (counts the starting quantity, then the items to be removed, then the remaining items).	Counts on 1, 2 or 3 from the first addend within 30. Counts back 1, 2, or 3 when minuend is 10 or fewer.	Counts on when the first addend is within 1-100 and the second addend is 1-5. Counts back 1-5 when the starting number is within 1-100. Can count up or back 1-5 to find the difference between two numbers 1-100 (e.g., 46 – 43)
Number Relationship Strategies					Mentally: adds 10 to a single-digit number adds a single-digit number to any decade number subtracts to a decade, e.g. 56 - 6 = 50. adds up to a decade, e.g. 48 + ☐ = 50 subtracts a single-digit number from a decade number, e.g. 60 - 4 = 56. Applies an informal understanding of: the commutative and associative properties of addition (i.e., does not always add numbers in the order in which they are presented.) the relationship between addition and subtraction (e.g.12 – 9 can also be thought of as 9 + ☐ = 12)
Composing/Decomposing			Can mentally compose and decompose numbers to 5 Can conceptually subitize quantities to 5	Can mentally compose and decompose numbers to 10 Can conceptually subitize quantities to 10	Can mentally compose and decompose numbers to 20
Fact Fluency				Fluent with addition and subtraction facts to 5, doubles to 5+5, and +1 facts	Fluent with addition facts to 10

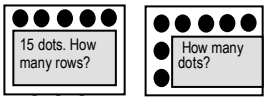
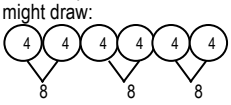
Developmental Guidelines

Operation Development – Addition and Subtraction Cont.

	Level 5	Level 6	Level 7	Level 8+
Common Core	1/2	2	2/3	3/4+
Problem Types	Uses strategies described below to solve story problems including: Add to: Result unknown and change unknown Take from: Result unknown and change unknown Put together/take apart: total unknown and addend unknown Compare: difference unknown (Appropriate number choices are described below.)	Uses strategies described below to solve story and number problems with numbers to 1,000 including all add to, take from, put together/take apart, and compare problem types. Uses strategies described below to solve story and number problems that involve more than two addends and/or multiple steps.		Uses strategies described below to solve story and number problems with numbers to 10,000 and beyond including all add to, take from, put together/take apart, and compare problem types. Uses strategies described below to solve story and number problems that involve more than two addends and/or multiple steps.
Strategies Supported by Materials	Can solve additive and subtractive covered tasks involving two-digit numbers modeled with base-10 materials	Can solve additive and subtractive covered tasks involving three-digit numbers when supported by base-10 materials		
Counting Strategies	Counts on when the first addend is within 1-1000 and the second addend is 1-5. Counts back 1-5 when the starting number is within 1-1000 Can count up or back 1-5 to find the difference between two numbers 1-1000 (e.g., 502 – 498) Counts on or back by 10s and 1s to add and subtract 2-digit numbers			
Number Relationship Strategies	Mentally: adds 10 to or subtracts 10 from any number 10-100. adds any decade to or subtracts any decade from numbers 10-100. e.g. 20 + 40, 56 + 30, 72 – 40. uses addition facts such as 4 + 3 to solve problems such as 54 + 3 and 40 + 30, and subtraction facts such as 7 - 5 to solve 57 – 5 and 70 - 50. decomposes a single-digit number to cross a decade when adding or subtracting within 100 e.g. 48 + 6 = (48 + 2) + 4 or 56 – 9 = (56 – 6) – 3 Applies an informal understanding of: the commutative and associative properties of addition the relationship between addition and subtraction	Has at least one viable strategy for solving addition and subtraction problems involving any 2-digit numbers (such as incrementing, tens-and-ones, or compensating). Can write down intermediate steps when needed to keep track of thinking. Mentally: adds 100 (or 200, 300, etc.) to, or subtracts 100 (or 200, 300, etc.) from any three-digit number 100-900 (e.g., 438 + 100, 350 – 200). finds the other part of 100 (e.g. 88 + ? = 100.) uses addition facts such as 7 + 8 to solve problems such as 157 + 8 or 70 + 80, and subtraction facts such as 14 – 9 to solve problems such as 314 – 9 or 140 – 90. decomposes a single-digit number to cross a hundred when adding or subtracting within 1,000 e.g., 497 + 5 = (497 + 3) + 2 Applies an informal understanding of: the commutative and associative properties of addition the relationship between addition and subtraction	Can **flexibly** choose among strategies such as incrementing, tens-and-ones, and compensating to solve addition and subtraction problems involving 2-digit numbers **efficiently** and **accurately.** Has at least one viable strategy for solving addition and subtraction problems involving any 3-digit numbers (such as incrementing, tens-and-ones, or compensating). Can write down intermediate steps when needed to keep track of thinking. Applies an informal understanding of: the commutative and associative properties of addition the relationship between addition and subtraction	Can **flexibly** choose among strategies such as incrementing, tens-and-ones, compensating, and other algorithms to solve addition and subtraction problems involving whole numbers **efficiently** and **accurately**. Can write down intermediate steps when needed to keep track of thinking. Applies an informal understanding of: the commutative and associative properties of addition the relationship between addition and subtraction Judges the reasonableness of results of addition and subtraction problems
Composing/Decomposing	Can mentally compose and decompose numbers to 100 into tens and ones in standard and non-standard ways (e.g. 74 is 70 + 4, 60 + 14, and 50 + 24)	Can mentally compose and decompose numbers to 1,000 in to hundreds, tens, and ones in standard and non-standard ways (e.g. 247 is 200 + 40 + 7, and 100 + 130 + 17)		Can mentally compose and decompose numbers to 10,000 and beyond in to thousands, hundreds, tens, and ones in standard and non-standard ways (e.g. 3,125 is 3,000 + 100 + 20 + 5, and 2,000 + 1,100 + 25)
Fact Fluency	Fluent with addition facts (with totals to 20) Fluent with subtraction facts (with totals to 10)	Fluent with some across-ten subtraction facts	Fluent with subtraction facts (with totals to 20)	

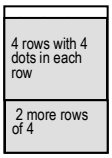
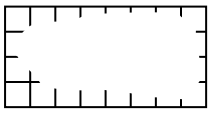
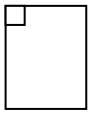
Developmental Guidelines

Operation Development – Multiplication and Division with whole numbers

	Level 0	Level 1	Level 2	Level 3	Level 4
Common Core	*Multiplication is not addressed in the Common Core State Standards before grade 2.*				
Problem Types	Solves Multiplication, Measurement Division and Partitive Division problems with totals to 5 embedded in daily routines **using direct modeling**.	Uses strategies described below to solve Multiplication, Measurement Division, and Partitive Division problems embedded in daily routines with totals to 10.	Uses strategies described below to solve Multiplication, Measurement Division, Partitive Division story problems with totals to 20.	Uses strategies described below to solve Multiplication, Measurement Division, and Partitive Division story problems with totals to 30.	Uses strategies described below to solve Multiplication, Measurement Division (MD), and Partitive Division (PD) story problems with totals to 50.
Strategies Based on Visual Models		Can describe, organize, and make equal groups and equal shares when supported by concrete objects. Objects at this level need to be fairly literal representations of the quantities in the problem (e.g. coats and buttons, plates and cookies, etc.) to help children differentiate between objects that represent the number of groups, and objects that represent the items in each group. Involving the child in acting out the problem may support the child's understanding of the problem.		Can describe, organize, and make equal groups and equal shares when supported by objects, drawings, or arrays. Objects used to model no longer need to be literal (buttons, cookies, etc.)	Can describe, organize, and make equal groups and equal shares when supported by objects, drawings, or arrays. Can determine the total number of dots, or number of groups, or number in each group in partially covered arrays. 
Direct Modeling and Counting Strategies		Counts by ones to determine answers, and does not pay attention to the structure of groups when counting. (In other words, once the child has modeled the problem, he or she sees all the items to be counted as a large collection of ones, rather than as several equal groups. Depending on the model that the child has created, he or she may or may not count one group, and then another group, and then another group. A child at this level who is trying to find out how many squares or dots are in a rectangular array will probably not count each row, or each column—he or she may count the items in a zigzag or spiral path.)		When counting the number of objects in a rectangular array or in groups, the child uses the structure of groups as they count. The child **may** use stress counting to count visible items in groups. Stress counting means counting by 1s, but stressing the multiples (e.g., 1, 2, **3**, 4, 5, **6**, 7, 8, **9**...).	Uses stress counting or skip counting (by 2s, 5s, or 10s) to count items in groups even when individual items are not visible, but relies on visible markers to keep track of the number of groups (e.g., if a child is told that there are 5 cookies on each plate and is then shown 3 plates, the child can figure out the total number of cookies even if the individual cookies aren't represented.)
Additive Strategies					**May** use repeated addition to solve multiplication problems. For example: for 6×4, the child might draw:  (Developmental Guidelines for Addition and Subtraction at this level show strategies that students are likely to use to find answers when using repeated addition.)

Developmental Guidelines

Operation Development – Multiplication and Division Cont.

	Level 5	Level 6	Level 7
Common Core	2/3	2/3	3/4
Problem Types	Uses strategies described below to solve: Multiplication story problems with factors up to 10×10 MD and PD story problems with dividends up to 50 and divisors up to 10. (May be able to solve MD and PD story problems with dividends to 100)	Uses strategies described below to solve Multiplication, Measurement Division, and Partitive Division story problems with totals to 100.	Uses strategies described below to solve: Multiplication story and number problems (with products of 1-digit x 2-digit numbers) MD and PD story and number problems (with dividends up to 100) Story problems that involve: multiple steps and multiple operations the interpretation of remainders
Strategies Based on Visual Models	Can combine two sets of equal groups when each set is covered:  Example: There are four rows with four dots in each row. How many dots is that? If I add two more rows of four, how many rows are there? How many dots altogether?	Can partition a rectangle into rows and columns of same-size squares and count to find the total number of them.  Example: If you finished covering this rectangle with squares, how many squares would there be altogether? Explores commutative property with arrays.	Can find the number of squares that would cover a rectangle when given the number of squares in a row and the number of squares in a column. Does not have to actually cover the space with squares to find the total.  Example: Four squares like the one shown can fit in a row across the top of this rectangle. Five squares fit down the side. How many squares would it take to cover the whole rectangle?
Direct Modeling and Counting Strategies	Can now think of a group as a unit that can be counted (i.e. "three ones" can now be thought of as "one three.") Uses stress counting or skip counting (by 2s, 3s, 4s, 5s, or 10s) to solve problems without needing a model to represent either the number of groups or the number in each group (e.g., to find out the number of cookies on 4 plates, with 3 cookies per plate, a child might count by 3s and keep track on fingers until she has made 4 counts).	**May** continue to model PD story problems, but may deal out in chunks. For example, for 80÷7, the child may get out 8 sets of 10 unifix cubes, then deal out 10s into 7 piles. Then, the child might break up the remaining 10 into ones and deal one more into each group, with 3 left over.	Initial strategies for multi-digit numbers may involve direct modeling by tens and ones.
Additive Strategies	Understands relationship between repeated addition and arrays (i.e., can write an addition number sentence to describe an array) Uses repeated addition to solve multiplication problems when the skip counting sequence is not known (writing all the groups out—example 7×8 = 8+8+8+8+8+8+8) Uses repeated addition or repeated subtraction to solve division problems (Developmental Guidelines for Addition and Subtraction at this level show strategies that students are likely to use to find answers when using repeated addition.)	Uses strategies such as repeated addition, repeated subtraction, repeated doubling, or other additive groupings to solve multiplication and division problems. For example, for 9 × 8, the child might write: 8+8+8 = 24 8+8+8 = 24 8+8+8 = 24 24+24+24 = 72 Later, the child may only write down the addends in one subgroup, and then make use of the subtotal multiple times: 8+8+8 = 24 24+24+24 = 72 (Developmental Guidelines for Addition and Subtraction at this level show strategies that students are likely to use to find answers when using repeated addition.)	Uses strategies such as repeated addition, repeated subtraction, repeated doubling, or other additive groupings to solve multiplication and division problems. For example, for 96÷16, the child might do the following: 16 +16 32 2 +32 64 4 +32 96 6 (Developmental Guidelines for Addition and Subtraction at this level show strategies that students are likely to use to find answers when using repeated addition.)
Multiplicative Strategies			Uses known facts: For 12×8, the child may do: 10×8 = 80 80+8+8 = 96 For 26÷8, the child might say, "3×8 is 24, so it's 3 with 2 left over.
Properties		Begins to informally apply the commutative property and distributive property. Note that students use strategies that employ the properties before they fully understand why the properties work. Begins to understand the inverse relationship between multiplication and division.	Applies an informal understanding of: the commutative property of multiplication the distributive property the relationship between multiplication and division
Fact Fluency		Fluent with multiplication facts for ×0, ×1, ×2, ×10, ×5 (including, for example, both 8×5 and 5×8)	Fluent with multiplication facts for ×0, ×1, ×2, ×10, ×5, ×3, ×4

Developmental Guidelines

Operation Development – Multiplication and Division Cont.

Level 8

Common Core: 3/4

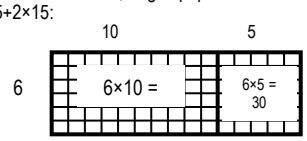
Problem Types

Uses strategies described below to solve:

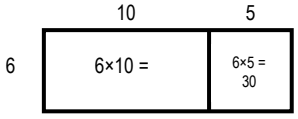
- Multiplication story and number problems (with products of 1-digit × 2-digit numbers)
- MD and PD story and number problems (with dividends up to 1,000 where either the divisor or the quotient is a single-digit number)
- Story problems that involve:
  - multiple steps and multiple operations
  - the interpretation of remainders

Strategies Based on Visual Models

Uses tiling, base-ten blocks, or grid paper to show that 6×15 is 6×10 + 6×5, or 2×15+2×15+2×15:



Begins to use open arrays:



Additive Strategies

Uses strategies such as repeated addition, repeated subtraction, repeated doubling, or other additive groupings to solve multiplication and division problems.

Multiplicative Strategies

Uses partial products for products of 1-digit x 2-digit numbers, especially where the 2-digit number is relatively small. For example, for 15×7:

10×7 = 70
5×7 = 35
70+35 = 105

(This strategy will often be supported by a drawing of an array.)

Solves division problems by using groups of the divisor to build up to the dividend (especially for Measurement Division), or by breaking the dividend into parts (especially for Partitive Division). For example, for 212÷8:

Groups of divisor	Breaking dividend apart
10×8 = 80 10×8 = 80 5×8 = 40 200 1×8 = 8 208 So the answer is 26 with 4 left over	200 is 8 groups of 25 You can put 1 more in each group. So there are 26 in each group with 4 left over.

Properties

Applies an informal understanding of:

- the commutative property of multiplication
- the distributive property
- the relationship between multiplication and division

Esti-mation

Fact Fluency

Fluent with multiplication facts for ×0, ×1, ×2, ×10, ×5, ×3, ×4

Fluent with some multiplication facts for ×6, ×7, ×8, ×9

Level 9

Common Core: 3/4

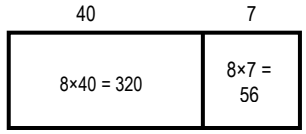
Problem Types

Uses strategies described below to solve:

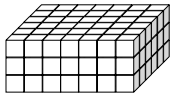
- Multiplication story and number problems (with products of 1-digit × 2-digit numbers and 1-digit × 3-digit numbers)
- MD and PD story and number problems (with dividends up to 1,000 where either the divisor or the quotient is a single-digit number)
- Story problems that involve:
  - multiple steps and multiple operations
  - the interpretation of remainders

Strategies Based on Visual Models

Uses open arrays to model multi-digit problems. For example, for 8×47:



Uses 3-D arrays to explore the associative property. For example, for 126 = (6×7)×3 = 6×(7×3)



Multiplicative Strategies

Mentally multiplies one-digit whole numbers by multiples of 10 in the range 10-90 (e.g., 9 × 80, 5 × 60)

Can **flexibly** choose among strategies such as partial products, making an equivalent problem (doubling and halving), or compensating to solve products of 1-digit × 2-digit and 1-digit × 3-digit numbers **efficiently** and **accurately**. For example:

Making an equivalent problem (Doubling and halving)	Compensating
For 35×8: 35×8 = 70×4 = 140×2 = 280	298×6 = 300×6-12 = 1,800-12 = 1,788

Solves division problems by using groups of the divisor to build up to the dividend, breaking the dividend into parts, making an equivalent problem, or compensating. For example:

Making an equivalent problem	Compensating
168÷24 = 84÷12 = 42÷6 = 21÷3 = 7	For 235÷80: 3×80 = 240, so 3 is too much. The answer is 2 with 75 left over.

Properties

Applies an informal understanding of:

- the commutative and associative properties of multiplication
- the distributive property
- the relationship between multiplication and division

Esti-mation

Judges the reasonableness of results of multiplication problems

Fact Fluency

Fluent with all multiplication facts to 10×10

Developmental Guidelines

Operation Development – Multiplication and Division Cont.

<table>
<tr><th></th><th>Level 10</th><th>Level 11</th></tr>
<tr><td>Common Core</td><td>4</td><td>5</td></tr>
<tr><td>Problem Types</td><td>Uses strategies described below to solve:
Multiplication story and number problems with products of:
1-digit number × a number with up to 4 digits
2-digit × 2-digit numbers
MD and PD story and number problems
with dividends up to 10,000 where either the divisor or the quotient is a single-digit number), or
with dividends up to 1,000 and divisors up to 100
Story problems that involve:
multiple steps and multiple operations
multiplicative comparisons
rates, proportions, ratios
combinations
the interpretation of remainders</td><td>Uses strategies described below to solve:
Multiplication story and number problems (with products up to 2-digit × 3-digit numbers)
MD and PD story and number problems
with dividends up to 10,000 where either the divisor or the quotient is a single-digit number), or
with dividends up to 1,000 and divisors up to 100
Story problems that involve:
multiple steps and multiple operations
multiplicative comparisons
rates, proportions, ratios
combinations
the interpretation of remainders</td></tr>
<tr><td>Strategies Based on Visual Models</td><td>Uses open arrays to model multi-digit problems. For example, for 32×48:
<table><tr><th></th><th>40</th><th>8</th></tr><tr><th>30</th><td>30×40 = 1,200</td><td>30×8 = 240</td></tr><tr><th>2</th><td>2×40 = 80</td><td>2×8 = 16</td></tr></table></td><td>Uses open arrays to model multi-digit problems. For example, for 46×376:
<table><tr><th></th><th>300</th><th>70</th><th>6</th></tr><tr><th>40</th><td>40×300 = 12,000</td><td>40×70 = 2,800</td><td>40×6 = 240</td></tr><tr><th>6</th><td>6×300 = 1,800</td><td>6×70 = 420</td><td>6×6 = 36</td></tr></table></td></tr>
<tr><td>Multiplicative Strategies</td><td>Mentally solves extended multiplication facts (e.g., 3×500, 8,000×7, 40×60)
Can flexibly choose among strategies such as partial products, making equivalent problems, or compensating to solve products of 1-digit × a number up to 4 digits efficiently and accurately.
Uses strategies such as partial products, making equivalent problems, compensating, and ratio tables to solve products of 2-digit×2-digit numbers. For example:
<table><tr><th>Partial Products</th><th>Making equivalent problems</th><th>Ratio table</th></tr><tr><td>For 32×48:
30×40 = 1,200
2×40 = 80
30×8 = 240
2×8 = 16
1,200+80+240 +16 = 1,536</td><td>For 24×12:
24×12 = 96×3 = 288
For 60×33:
60×33 = 6×10×33 = 6×330 = 1,980</td><td>For 24×12:<table><tr><td>1</td><td>10</td><td>20</td><td>4</td><td>24</td></tr><tr><td>12</td><td>120</td><td>240</td><td>48</td><td>288</td></tr></table></td></tr></table>Solves division problems by using groups of the divisor to build up to the dividend, breaking the dividend into parts, making an equivalent problem, or compensating.</td><td>Can flexibly choose among strategies such as partial products, making equivalent problems, compensating, ratio tables, and other algorithms to solve 2-digit × 2-digit numbers efficiently and accurately.
Uses strategies such as partial products, making equivalent problems, compensating, ratio tables, and other algorithms to solve products of 2-digit × 3-digit numbers.
Can flexibly choose among strategies such as using groups of the divisor to build up to the dividend, breaking the dividend into parts, making an equivalent problem or compensating to efficiently and accurately solve division problems involving dividends up to 10,000 where either the divisor or the quotient is a single-digit number.
Solves division problems involving dividends up to 1,000 and divisors up to 100 by using groups of the divisor to build up to the dividend, breaking the dividend into parts, making an equivalent problem or compensating.</td></tr>
<tr><td>Properties</td><td>Applies an informal understanding of:
the commutative and associative properties of multiplication
the distributive property
the relationship between multiplication and division</td><td>Applies an informal understanding of:
the commutative and associative properties of multiplication
the distributive property
the relationship between multiplication and division</td></tr>
<tr><td>Esti-mation</td><td>Judges the reasonableness of results of multiplication problems</td><td>Judges the reasonableness of results of multiplication problems</td></tr>
<tr><td>Composing/Decomposing</td><td>Can:
find all factor pairs for a whole number in the range 1-100
determine whether a given whole number in the range 1-100 is a multiple of a given one-digit number.
determine whether a given whole number in the range 1-100 is prime or composite.
Recognizes that in a multi-digit whole number, a digit in one place represents ten times what it represents in the place to its right. For example, recognizes that 700 ÷ 70 = 10 by applying concepts of place value and division.</td><td>Can find the least common multiple or greatest common factor of any pair of numbers in the range 1-100.
Can generalize about patterns in the number of zeros of the product when multiplying a number by powers of 10.
Recognizes that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left. For example, recognizes 8,000 is 1/10 of 80,000.</td></tr>
<tr><td>Fact Fluency</td><td>Fluent with some division facts</td><td>Fluent with all division facts</td></tr>
</table>

Operation Development – Multiplication and Division Cont.

	Level 12+
Common Core	5/6
Problem Types	Uses strategies described below to solve: Multiplication, MD and PD story and number problems with totals to 10,000 and beyond Story problems that involve: multiple steps and multiple operations multiplicative comparisons rates, proportions, ratios combinations the interpretation of remainders
Multiplicative Strategies	Can **flexibly** choose among strategies such as partial products, making equivalent problems, compensating, ratio tables, and other algorithms to solve products of 2-digit × 3-digit numbers **efficiently** and **accurately**. Can **flexibly** choose among strategies such as using groups of the divisor to build up to the dividend, breaking the dividend into parts, making an equivalent problem or compensating to **efficiently** and **accurately** solve division problems involving dividends up to 1,000 and divisors up to 100 (For multiplication and division problems involving numbers larger than these, the most efficient strategy **may** involve estimating the magnitude of the answer and using a calculator to perform the computation.)
Properties	Applies an informal understanding of: the commutative and associative properties of multiplication the distributive property the relationship between multiplication and division
Estimation	Judges the reasonableness of results of multiplication and division problems
Composing/ Decomposing	Can find the prime factorization for numbers in the range 1-100 Can express the prime factorization of numbers using exponential notation. Can generalize about patterns in the number of zeros of the product when multiplying numbers involving powers of 10. Uses whole-number exponents to denote powers of 10.

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