SPATIAL REASONING WITH OPEN-ENDED TASKS: THE WET SHIPPING CRATE

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Subtly hidden within our mathematics standards, usually mentioned in passing in the context of the geometry and measurement strand, is a critical mathematical competency that I encounter daily in my world. Whether it's stashing leftovers in the refrigerator, building and filling a raised bed garden in the backyard, or finding space in the pantry for that newly designed granola bar packaging, spatial reasoning is a part of our lives every day. More importantly, fluency with spatial reasoning is critical in a wide variety of domains from electrical circuit design to gaming and these competencies can be both taught and learned (National Research Council, 2006). Technological developments in our 21st-century world such as geographic information systems (GIS) place increasing demands on both everyday and expert spatial reasoning skills (Ontario Ministry of Education, 2014). This raises the question – in what ways are we, and can we, foster spatial reasoning competencies in our students?

The most popular tasks related to spatial reasoning are closed-ended multiple choice visualization tasks, embodied by items from the Vandenberg Kuse Mental Rotations Test as shown in Figure 1. Such items present an isometric drawing of a three-dimensional shape and ask students to determine which of the rotated figures is a representation of the same shape. These items have long been used to assess students' spatial reasoning and conduct research, but they do not constitute strong instructional tools for the classroom. In this article, I share an example of an open-ended spatial reasoning task, The Wet Shipping Crate, with multiple solution paths that can be used to foster rich classroom discourse around spatial reasoning.

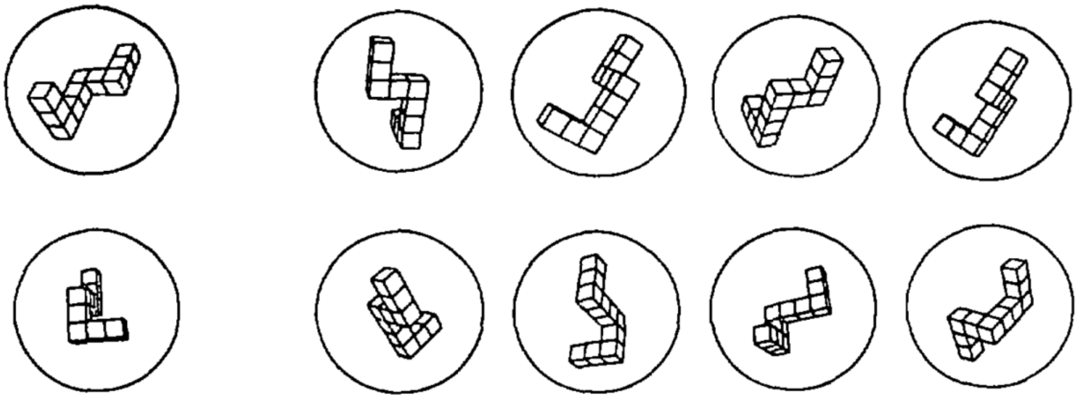

Figure 1. Mental Rotations Test item (Vanderberg & Kuse, 1978).

The Nature of Spatial Reasoning

The National Research Council's (2006) landmark synthesis of spatial reasoning notes three key components: "concepts of space, tools of representation, and processes of reasoning" (p. 3). Concepts of space relates to students' abilities to visualize figures, to recognize properties of figures that remain invariant as they are transformed and reoriented, and to orient shapes and solids in broader contexts. Tools of representation include physical, drawn, and digital representations of 2-D and 3-D objects that facilitate spatial visualization and reasoning. Too often, spatial reasoning work in PreK-12 spaces relies heavily on drawn 2-D representations of objects rather than a robust array of representations that can facilitate flexible reasoning. Processes of reasoning involve the ability to mobilize geometric properties and make oral and written arguments that justify spatial reasoning, such as being able to identify and explain the three different altitudes of a triangle and justify the ways in which they represent an important quantity related to the triangle's area.

While PreK-12 academic standards include explicit attention to geometry and measurement, they rarely identify spatial reasoning as an important consideration within the study of geometry and measurement. For example, the Wisconsin Standards for Mathematics (Wisconsin Department of Public

Instruction, 2021) only include spatial reasoning explicitly in discussing the nature of mathematical study in kindergarten. However, a number of key ideas in geometry and measurement standards have strong connections to spatial reasoning. NCTM's Grades 6-8 Essential Understandings in Geometry and Measurement contain two big ideas that directly implicate spatial reasoning: Geometric thinking involved developing, attending to, and learning how to work with imagery (Big Idea 2), and A geometric object is a mental object that, when constructed, carries traces of the tool or tools by which it was constructed (Big Idea 3); additionally, the fourth big idea related to justifying in geometric investigations has a strong relationship to spatial reasoning (Sinclair, Pimm, & Skelin, 2012).

Wisconsin's middle and high school standards also present a number of opportunities for students to grapple with aspects of spatial reasoning in the service of geometry and measurement standards. Many of the Wisconsin Mathematics Standards in grades 6-12 provide opportunities for teachers to use spatial reasoning-focused tasks (see Appendix A for a list). Addressing spatial reasoning can also take place in the context of more open-ended tasks as compared to the ones shown in Figure 1. As with other areas of mathematics, spatial reasoning tasks should involve complex, non-algorithmic tasks referred to as doing mathematics tasks (Smith & Stein 1998) that provide students with opportunities to engage in rich mathematics discourse, to create mathematical arguments and defend those arguments (Smith, Steele, & Raith, 2017), and to use mathematical modeling processes (Consortium for Mathematics and its Applications & Society for Industrial and Applied Mathematics, 2019). The Wet Shipping Crate task is an example of such a task, designed to support spatial reasoning and address geometry and measurement content, afford the use of multiple mathematical representations, and provide multiple non-algorithmic solution paths.

Table 1. Wisconsin Geometry and Measurement Standards associated with spatial reasoning

Level	Notation	Standard
6	M.6.G.A.1	Find the area of right triangles, other triangles, special quadrilaterals, and polygons by composing into rectangles or decomposing into triangles and other shapes; apply these techniques in the context of solving real-world and mathematical problems.
6	M.6.G.A.2	Find volumes of right rectangular prisms with fractional edge lengths by using physical or virtual unit cubes. Develop (construct) and apply the formulas $V = l\,w\,h$ and $V = B\,h$ to find volumes of right rectangular prisms in the context of solving real-world and mathematical problems.
6	M.6.G.A.4	Represent three-dimensional figures using nets made up of rectangles and triangles, and use the nets to find the surface area of these figures. Apply these techniques in the context of solving real-world and mathematical problems.
7	M.7.G.A.1	Solve problems involving scale drawings of geometric figures, including computing actual lengths and areas from a scale drawing and reproducing a scale drawing at a different scale.
7	M.7.G.A.2	Draw (freehand, with ruler and protractor, and with technology) geometric shapes with given conditions. Focus on constructing triangles from three measures of angles or sides, noticing when the conditions determine a unique triangle, more than one triangle, or no triangle.
7	M.7.G.A.3	Describe the two-dimensional figures that result from slicing three dimensional figures parallel to the base, as in plane sections of right rectangular prisms and right rectangular pyramids.
7	M.7.G.B.6	Solve real-world and mathematical problems involving area, volume and surface area of two- and three-dimensional objects composed of triangles, quadrilaterals, polygons, cubes, and right prisms.
8	M.8.G.A.1	Verify experimentally the properties of rotations, reflections, and translations: a. Lines are taken to lines, and line segments to line segments of the same length. b. Angles are taken to angles of the same measure. c. Parallel lines are taken to parallel lines.
8	M.8.G.A.2	Understand that a two-dimensional figure is congruent to another if the second can be obtained from the first by a sequence of rotations, reflections, and translations; given two congruent figures, describe a sequence that exhibits the congruence between them.
8	M.8.G.A.3	Describe the effect of dilations, translations, rotations, and reflections on two-dimensional figures using coordinates.
8	M.8.G.A.4	Understand that a two-dimensional figure is similar to another if the second can be obtained from the first by a sequence of rotations, reflections, translations, and dilations; given two similar two-dimensional figures, describe a sequence that exhibits the similarity between them.
8	M.8.G.C.9	Know the relationship among the formulas for the volumes of cones, cylinders, and spheres (given the same height and diameter) and use them to solve real-world and mathematical problems.
HS	M.G.SRT.A.2	Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.
HS	M.G.GMD.C .4	Use geometric shapes, their measures, and their properties to describe objects (e.g., modeling a tree trunk or a human torso as a cylinder).
HS	M.G.GMD.C .6	Apply geometric methods to solve design problems (e.g., designing an object or structure to satisfy physical constraints or minimize cost; working with typographic grid systems based on ratios).

The Wet Shipping Crate Task

The Wet Shipping Crate task was developed for a course in middle grades geometry and measurement for preservice and practicing teachers and I have used the task with both students and teachers. I have adapted the task over the years to keep the contexts contemporary while providing opportunities to engage with important spatial reasoning ideas. In the contexts of my use of the task with students, I see the task as primarily addressing Wisconsin mathematics standards M.G.GMD.C.6 and the modeling standards. In the context of work with teachers, I see the task as an opportunity to discuss the NCTM Effective Teaching Practices use and connect mathematical representations, facilitate meaningful mathematics discourse, support productive struggle, and implement tasks that promote reasoning and problem solving (NCTM, 2014).

<u>Part 1</u>
A shipper is shipping large crate of iPhones from Shanghai. Each large crate measures 36”x18”x16”. It is filled completely with iPhones, which are in boxes that are 3”x4”x6”.

One crate falls off the boat and submerges completely but is pulled out of the water almost immediately.

Upon talking with Apple, the shipper concludes that all iPhones that were touching the outside of the wet crate will have to be returned to Apple to check if they still work. How many iPhones will have to go back? Be sure to explain how you arrived at your answer, and use words, symbols, and/or diagrams to support your explanation.

<u>Part 2</u>
Upon hearing the bad news, Apple decides that they need to ship their iPhones in a larger crate. They want to design a crate that has twice the volume of the original and that minimizes the number of iPhones that would be damaged in a similar accident.

What are the dimensions of a crate that serves this purpose?

If Apple’s crate supplier charges by the square inch of surface area, how much more will the new crate cost?

Figure 2. The Wet Shipping Crate Task, Parts 1 and 2, adapted from Steele (2006).

Pause for a moment and consider how you might solve the task. In the next section, I'll share how students and teachers have approached the task and how those approaches provide interesting opportunities for spatial reasoning.

Solution Paths for The Wet Shipping Crate

There are a number of possible solutions and solution paths to the task. If you worked the problem, you might have started down a road that didn't end up being productive or noticed that there were some key decision points in which you could make modeling assumptions that would potentially change the outcome.

To solve Part 1, many students and teachers begin the task by drawing a three-dimensional representation of the crate like the one shown in Figure 3. They then proceed to count the number of phones on each face, noting that some of the iPhone boxes inside have faces that intersect multiple faces of the larger shipping container. As a result, one has to keep track of the faces of an iPhone box that are double- or triple-counted and subtract them back out.

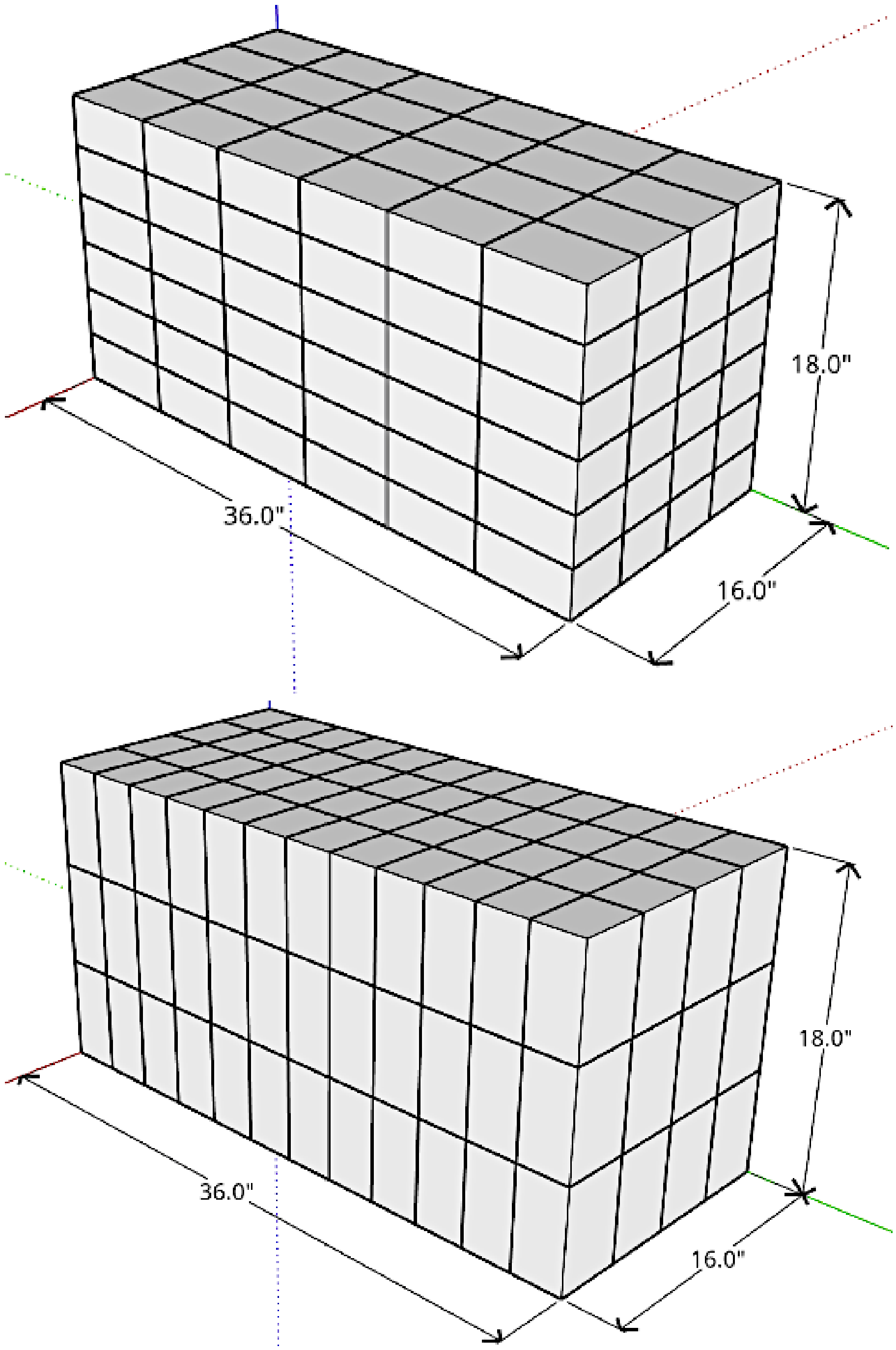

Figure 3. Isometric representations for the Wet Shipping Box.

Another approach is to consider the crate as a series of layers, the top and bottom of which would be fully impacted by the submerged shipping crate and the middle layers of which would feature a ring of iPhones around the outside that are impacted. Figure 4 shows one example of such a solution, with each layer of the shipping crate drawn.

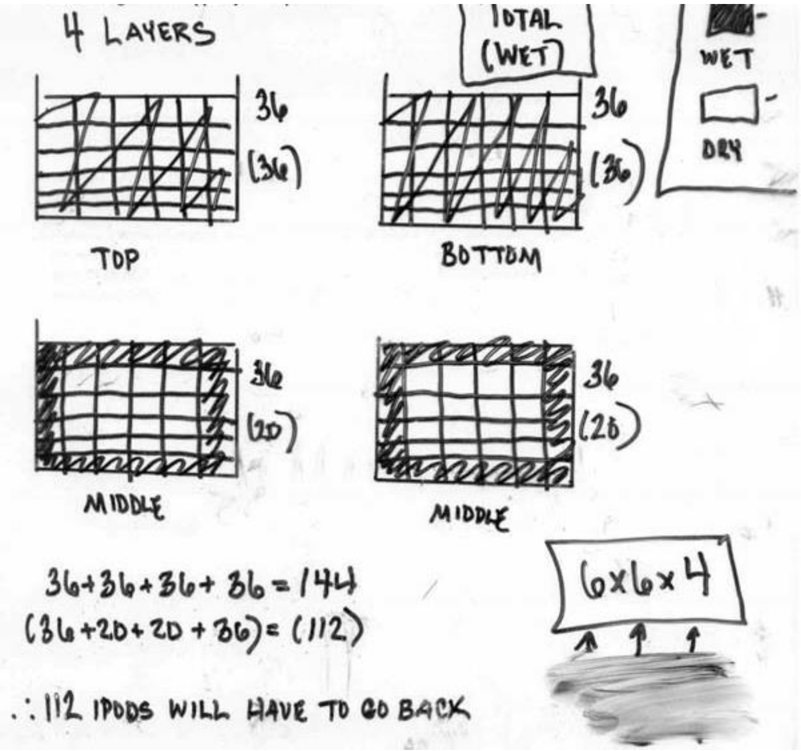

Figure 4. Layer-based spatial reasoning to solve the Wet Shipping Crate task (Steele, 2006).

A third common pathway for Part 1 is to approach the task strictly numerically. One can calculate the entire volume of the shipping container (36x18x16=10,368 in^3) and divide by the volume of the iPhone box (3x4x6=72 in^3) to determine that there are 144 iPhones that will fit in the crate. While this may be the fastest way to arrive at the total number of iPhones in the shipping crate, it doesn't address the issue of how they are arranged and how to determine how many may have gotten wet. This compels learners to engage with the spatial reasoning aspects of the task.

For Part 1, you may have noticed that there are multiple correct answers depending on how one arranges the phones in the shipping crate. Depending on how the phones are arranged, answers of 112, 124, and 94 are all defensible.

Part 2 provides opportunities for learners to connect the spatial reasoning to a conceptual understanding of volume as well as the formula. The most common approach to doubling the size of the shipping crate is to double the length of a side, which connects nicely to the V-lwh volume formula for a rectangular prism. There are, however, additional configurations that double the number of iPhones in the shipping container while further minimizing the numbers on the edges. Learners then need to coordinate two- and three-dimensional considerations to determine the impact on materials costs.

Opportunities for Spatial Reasoning

The Wet Shipping Crate task is rich with opportunities for learners to engage in spatial reasoning using a meaningful contextual situation. In discussing Part 1, learners can make use of orthographic and isometric drawings, digital renderings, and physical manipulatives to describe how they decided to pack the shipping container and how they made sense of counting the number of iPhone boxes that were potentially damaged in the accident. Describing those choices requires being able to visualize how the box is put together, choose a representation that is useful both to convey the visualization and to conduct the count, and to explain their thinking in a way that makes sense to other learners. Often, the representation used to decide how the box is packed is different from the representation a learner or group uses to count, requiring the coordination of multiple representations in addition to the symbolic work needed to keep count. When enacting this task, a great deal of the discourse often centers on which representations were useful for which purpose, choices that were made and abandoned in the process of productive struggle, and the ways in which learners coordinated 2-D and 3-D renderings of the box. Part 2 requires the coordination of two- and three-dimensional measurements (surface area and volume), understanding the relationship between them, and making use of those relationships in ways that go beyond simply substituting values and evaluating a formula. The assumptions and choices learners make represent key elements of the mathematical modeling cycle and present opportunities for learners to make choices, iterate their models, revise their assumptions, and to test if their model has improved.

Discussion

Spatial reasoning is a critical mathematical competency that is often hidden or neglected entirely in our standards and curriculum materials. Often, the only opportunities students have to work spatial reasoning tasks consist of closed-form multiple choice items on a standardized assessment. As the Wet Shipping Crate task demonstrates, students can be provided with meaningful opportunities to engage in open-ended spatial reasoning tasks that afford multiple solution pathways and opportunities for productive discourse. Moreover, these tasks can address multiple geometry and measurement standards in deep, conceptual ways, in this case through the use of surface area and volume constructs to engage in the mathematical modeling cycle.

The teaching of geometry in middle and high school has in the past been characterized by memorization and rote application of formulas, alongside the repetition of procedural content across multiple grade levels. I encourage teachers to take up the notion of spatial reasoning as a vector to open up the content of geometry and measurement to the incorporation of more rich tasks, mathematical discourse, productive struggle, and the mathematical modeling cycle.

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